Journal of Statistical Physics, Vol. 113, Nos. 3/4, November 2003 (© 2003)

Janossy Densities. II. Pfaffian Ensembles

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Received November 14, 2002; accepted May 8, 2003

We extend the main result of the companion paper *J. Stat. Phys.* **113**:595–610 to the case of the pfaffian ensembles.

KEY WORDS: Random matrices; orthogonal polynomials; Janossy densities; pfaffian ensembles.

1. INTRODUCTION AND FORMULATION OF RESULTS

Let us consider a $2n$-particle pfaffian ensemble introduced by Rains in ref. 9: Let (X, λ) be a measure space, $\phi_1, \phi_2,..., \phi_{2n}$ be complex-valued functions on X, and $\epsilon(x, y)$ be an antisymmetric kernel such that

$$p(x_1,..., x_{2n}) = (1/Z_{2n}) \det(\phi_j(x_k))_{j,\, k=1,..., 2n}\, pf(\epsilon(x_j, x_k))_{j,\, k=1,..., 2n} \tag{1}$$

defines the density of a $2n$-dimensional probability distribution on $X^{2n} = X \times \cdots \times X$ with respect to the product measure $\lambda^{\otimes 2n}$. Ensembles of this form were introduced in refs. 9 and 11. We recall (see, e.g., ref. 5) that the pfaffian of a $2n \times 2n$ antisymmetric matrix $A = (a_{jk})$, $j, k = 1,..., 2n$, $a_{jk} = -a_{kj}$, is defined as $pf(A) = \sum_\tau (-1)^{\mathrm{sign}(\tau)} a_{i_1 j_1} \cdots a_{i_n, j_n}$, where the summation is over all partitions of the set $\{1,..., 2m\}$ into disjoint pairs $\{i_1, j_1\},..., \{i_n, j_n\}$ such that $i_k < j_k$, $k = 1,..., n$, and $\mathrm{sign}(\tau)$ is the sign of the permutation $(i_1, j_1,..., i_n, j_n)$. The normalization constant in (1) (usually called the partition function)

$$Z_{2n} = \int_{X^{2n}} \det(\phi_j(x_k))_{j,\, k=1,..., 2n}\, pf(\epsilon(x_j, x_k))_{j,\, k=1,..., 2n} \tag{2}$$

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0022-4715/03/1100-0611/0

can be shown to be equal to $(2n)!\, pf(M)$, where the $2n \times 2n$ antisymmetric matrix $M = (M_{jk})_{j,\, k=1,\dots,\, 2n}$ is defined as

$$M_{jk} = \int_{X^2} \phi_j(x)\, \epsilon(x, y)\, \phi_k(y)\, \lambda(dx)\, \lambda(dy). \tag{3}$$

For the pfaffian ensemble (1) one can explicitly calculate k-point correlation functions

$$\rho_k(x_1,\dots, x_k) := ((2n)!/(2n-k)!) \times \int_{X^{2n-k}} p(x_1,\dots, x_k, x_{k+1},\dots, x_{2n})\, d\lambda(x_{k+1}) \cdots d\lambda(x_{2n}), \quad k = 1,\dots, 2n$$

and show that they have the pfaffian form (ref. 9)

$$\rho_k(x_1,\dots, x_k) = pf(K(x_i, x_j))_{i,\, j=1,\dots,\, k}, \tag{4}$$

where $K(x, y)$ is the antisymmetric matrix kernel

$$K(x, y) = \begin{pmatrix} \sum_{1 \leqslant j, k \leqslant 2n} \phi_j(x)\, M_{jk}^{-t} \phi_k(y) & \sum_{1 \leqslant j, k \leqslant 2n} \phi_j(x)\, M_{jk}^{-t} (\epsilon\phi_k)(y) \\ \sum_{1 \leqslant j, k \leqslant 2n} (\epsilon\phi_j)(x)\, M_{jk}^{-t} \phi_k(y) & -\epsilon(x, y) + \sum_{1 \leqslant j, k \leqslant 2n} (\epsilon\phi_j)(x)\, M_{jk}^{-t} (\epsilon\phi_k)(y) \end{pmatrix}, \tag{5}$$

provided the matrix M is invertible (by definition $(\epsilon\phi)(x) = \int_X \epsilon(x, y)\, \phi(y)\, \lambda(dy)$). If $X \subset \mathbb{R}$ and λ is absolutely continuous with respect to the Lebesgue measure, then the probabilistic meaning of the k-point correlation functions is that of the density of probability to find an eigenvalue in each infinitesimal interval around points $x_1, x_2,\dots, x_k$. In other words

$$\rho_k(x_1, x_2,\dots, x_k)\, \lambda(dx_1) \cdots \lambda(dx_k) = \Pr\{\text{there is a particle in each infinitesimal interval } (x_i, x_i + dx_i)\}.$$

On the other hand, if μ is supported by a discrete set of points, then

$$\rho_k(x_1, x_2,\dots, x_k)\, \lambda(x_1) \cdots \lambda(x_k) = \Pr\{\text{there is a particle at each of the points } x_i,\ i = 1,\dots, k\}.$$

In general, random point processes with the k-point correlation functions of the pfaffian form (4) are called pfaffian random point processes.[8] Pfaffian point processes include determinantal point processes[10] as a particular case when the matrix kernel has the form $\begin{pmatrix} \epsilon & K \\ -K & 0 \end{pmatrix}$ where K is a scalar kernel and ϵ is an antisymmetric kernel.

So-called Janossy densities $\mathcal{J}_{k,I}(x_1,\dots, x_k)$, $k = 0, 1, 2,\dots$, describe the distribution of the eigenvalues in any given interval I. If $X \subset \mathbb{R}$ and λ is absolutely continuous with respect to the Lebesgue measure then

$$\mathcal{J}_{k,I}(x_1,\dots, x_k)\, \lambda(dx_1)\cdots\lambda(dx_k)$$
$$= \Pr\{\text{there are exactly } k \text{ particles in } I, \text{ one in each of the } k \text{ distinct infinitesimal intervals } (x_i, x_i + dx_i)\}.$$

If λ is discrete then

$$\mathcal{J}_{k,I}(x_1,\dots x_k)$$
$$= \Pr\{\text{there are exactly } k \text{ particles in } I, \text{ one at each of the } k \text{ points } x_i,\ i = 1,\dots, k\}.$$

See refs. 3 and 4 for details and additional discussion. For pfaffian point processes the Janossy densities also have the pfaffian form (see refs. 8 and 9) with an antisymmetric matrix kernel L_I:

$$\mathcal{J}_{k,I}(x_1,\dots, x_k) = \mathrm{const}(I)\, pf(L_I(x_i, x_j))_{i,j=1,\dots,k}, \tag{6}$$

where

$$L_I = K_I(Id + JK_I)^{-1}, \tag{7}$$

$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathrm{const}(I) = pf(J - K_I)$ is the Fredholm pfaffian of the restriction of the operator K on the interval I, i.e., $\mathrm{const}(I) = pf(J - K_I)$ $= (pf(J + L_I))^{-1} = (\det(Id + JK_I))^{1/2} = (\det(Id - JL_I))^{-1/2}$. (We refer the reader to ref. 9, Section 8 for the treatment of Fredholm pfaffians).

Let us define three $2n \times 2n$ matrices G^I, M^I, $M^{X\setminus I}$:

$$G^I_{jk} = \int_I \phi_j(x) \int_X \epsilon(x, y)\, \phi_k(y)\, \lambda(dy)\, \lambda(dx), \tag{8}$$

$$M^I_{jk} = \int_{I^2} \phi_j(x)\, \epsilon(x, y)\, \phi_k(y)\, \lambda(dx)\, \lambda(dy), \tag{9}$$

$$M^{X\setminus I}_{jk} = \int_{(X\setminus I)^2} \phi_j(x)\, \epsilon(x, y)\, \phi_k(y)\, \lambda(dx)\, \lambda(dy) \tag{10}$$

(please compare (9) and (10) with the above formula (3) for M). Throughout the paper we will assume that the matrices M^I and $M^{X\setminus I}$ are invertible.

The main result of this paper is

Theorem 1.1. The kernel L_I has a form similar to the formula (5) for K. Namely, L_I is equal to

$$L_I(x,y)=\begin{pmatrix} \sum_{1\leqslant j,k\leqslant 2n} \phi_j(x)(M^{X\setminus I})^{-t}_{jk}\,\phi_k(y) & \sum_{1\leqslant j,k\leqslant 2n} \phi_j(x)(M^{X\setminus I})^{-t}_{jk}\,(\epsilon_{X\setminus I}\phi_k)(y) \\ \sum_{1\leqslant j,k\leqslant 2n} (\epsilon_{X\setminus I}\phi_j)(x)(M^{X\setminus I})^{-t}_{jk}\,\phi_k(y) & -\epsilon_{X\setminus I}(x,y)+\sum_{1\leqslant j,k\leqslant 2n} (\epsilon_{X\setminus I}\phi_j)(x)(M^{X\setminus I})^{-t}_{jk}\,(\epsilon_{X\setminus I}\phi_k)(y) \end{pmatrix}, \tag{11}$$

where $\epsilon_{X\setminus I}\phi(x)=\int_{X\setminus I}\epsilon(x,y)^{+}\,\phi(y)\,\lambda(dy)$.

Comparing (11) with (5) one can see that the kernel L_I is constructed in the following way: 1) first it is constructed on $X\setminus I$ by the same recipe used to construct the kernel K on the whole X; 2) it is extended then to I (we recall that L_I acts on $L^2(I, d\lambda(x))$, not on $L^2(X\setminus I, d\lambda(x))$).

This result contains as a special case Theorem 1.1 from the companion paper.[3] The rest of the paper is organized as follows. We discuss some interesting special cases of the theorem, namely so-called polynomial ensembles ($\beta=1$, 2, and 4) in Section 2. The proof of the theorem is given in Section 3.

2. RANDOM MATRIX ENSEMBLES WITH $\beta=1, 2, 4$

We follow the discussion in ref. 9 (see also refs. 11 and 12).

Biorthogonal Ensembles

Consider the particle space to be the union of two identical measure spaces (V,μ) and (W,μ): $X=V\cup W, V=W$. The configuration of $2n$ particles in X will consist of n particles $v_1,\ldots,v_n$ in V and n particles $w_1,\ldots,w_n$ in W in such a way that the configurations of particles in V and W are identical (i.e., $v_j=w_j$, $j=1,\ldots,n$). Let ξ_j, ψ_j, $j=1,\ldots,n$ be some functions on V. We define $\{\phi_j\}$ and ϵ in (1) so that $\phi_j(v)=0$, $v\in V$, $\phi_j(w)=\xi_j(w)$, $w\in W$, $j=1,\ldots,n$, $\phi_j(v)=\psi_{j-n-1}(v)$, $v\in V$, $\phi_j(w)=0$, $w\in W$, $j=n+1,\ldots,2n$, and $\epsilon(v_1,v_2)=0$, $v_1,v_2\in V$, $\epsilon(w_1,w_2)=0$, $w_1,w_2\in W$, $\epsilon(v,w)=-\epsilon(w,v)=\delta_{vw}$, $v\in V, w\in W$. The restriction of the measure λ on both V and W is defined to be equal to μ. Then (1) specializes into (see Corollary 1.5. in ref. 9)

$$p(v_1,..., v_n) = \text{const}_n \det(\xi_j(v_i))_{i,j=1,...,n} \det(\psi_j(v_i))_{i,j=1,...,n}. \tag{12}$$

Ensembles of the form (12) are known as biorthogonal ensembles (see refs. 1 and 7). The statement of the Theorem 1.1 in the case (12) has been proven in the companion paper.[3] The special case of the biorthogonal ensemble (12) when $V = \mathbb{R}$, $\xi_j(x) = \psi_j(x) = x^{j-1}$, and $V = \{\mathbb{C} \mid |z| = 1\}$, $\xi_j(z) = \bar{\psi}_j(z) = z^{j-1}$, such ensembles are well known in Random Matrix Theory as *unitary ensembles*, see ref. 6 for details. An ensemble of the form (12) which is different from random matrix ensembles was studied in ref. 7. We specifically want to single out the polynomial ensemble with $\beta = 2$.

Polynomial (β=2) *Ensembles*

Let $X = \mathbb{R}$ or $\mathbb{Z}$, $\phi_j(x) = x^{j-1}$, $j = 1,..., 2n$, and $\lambda(dx)$ has a density $\omega(x)$ with respect to the reference measure on X (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (12) specializes into

$$p(v_1,..., v_n) = \text{const}_n \prod_{1 \leqslant i < j \leqslant n} (v_i - v_j)^2 \prod_{1 \leqslant j \leqslant n} \omega(v_j). \tag{13}$$

The next two ensembles we want to mention are the polynomial $\beta = 1$ and 4 ensembles.

Polynomial (β=1) *Ensembles*

Let $X = \mathbb{R}$ or $\mathbb{Z}$, $\phi_j(x) = x^{j-1}$, $j = 1,..., 2n$, $\epsilon(x, y) = \frac{1}{2} \operatorname{sgn}(y - x)$, and $\lambda(dx)$ has a density $\omega(x)$ with respect to the reference measure on X (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (1) specializes into the formula for the density of the joint distribution of $2n$ particles in a so-called $\beta = 1$ polynomial ensemble (see ref. 9, Remark 1):

$$p(x_1,..., x_{2n}) = \text{const}_n \prod_{1 \leqslant i < j \leqslant 2n} |x_i - x_j| \prod_{1 \leqslant j \leqslant 2n} \omega(x_j). \tag{14}$$

In Random Matrix Theory the ensembles (14) in the continuous case are known as *orthogonal ensembles* , see ref. 6.

Polynomial (β=4) *Ensembles*

Similar to the biorthogonal case ($\beta = 2$) let us consider the particle space to be the union of two identical measure spaces (Y, μ), (Z, μ), $X = Y \cup Z$, $Y = Z$, where $Y = \mathbb{R}$ or $Y = \mathbb{Z}$. The configuration of $2n$ particles $x_1,..., x_{2n}$, in X will consist of n particles $y_1,..., y_n$ in Y

and n particles $z_1,\dots, z_n$, in Z in such a way that the configurations of particles in Y and Z are identical. We define $\{\phi_j\}$ and ϵ so that $\phi_j(y)=y^j$, $y\in Y$, $\phi_j(z)=jz^{j-1}$, $z\in Z$, $\epsilon(y_1,y_2)=0$, $\epsilon(z_1,z_2)=0$, $\epsilon(y,z)=-\epsilon(z,y)=\delta_{yz}$. As above we assume that the measure μ has a density ω with respect to the reference measure on Y. Then the formula (1) specializes into the formula for the density of the joint distribution of n particles in a $\beta=4$ polynomial ensemble (see Corollary 1.3. in ref. 9)

$$p(y_1,\dots, y_n)=\mathrm{const}_n \prod_{1\leqslant i<j\leqslant n} (y_i-y_j)^4 \prod_{1\leqslant j\leqslant n} \omega(y_j). \tag{15}$$

In Random Matrix Theory the ensembles (15) are known as *symplectic ensembles*, see ref. 6.

3. PROOF OF THE MAIN RESULT

Consider matrix kernels

$$\mathscr{K}_I=-JK_I, \qquad \mathscr{L}_I=-JL_I. \tag{16}$$

Then the relation (7) simplifies into

$$\mathscr{L}_I=\mathscr{K}_I(\mathrm{Id}-\mathscr{K}_I)^{-1} \tag{17}$$

which is the same relation that is satisfied by the correlation and Janossy scalar kernels in the determinantal case.[2,4] The consideration of $\mathscr{K}_I$ and $\mathscr{L}_I$ is motivated by the fact that the pfaffians of the $2k\times 2k$ matrices with the antisymmetric matrix kernels K_I and L_I are equal to the quaternion determinants[6] of $2k\times 2k$ matrices with the kernels $\mathscr{K}_I$, $\mathscr{L}_I$ when the latter matrices are viewed as $k\times k$ quaternion matrices (i.e., each quaternion entry corresponds to a 2×2 block with complex entries). It follows from (5) and (16) that the kernel $\mathscr{K}_I$ is given by the formula

$$\mathscr{K}_I=\sum_{j,k=1,\dots,2n} M^{-t}_{jk}\begin{pmatrix} -(\epsilon\phi_j)\otimes\phi_k & -(\epsilon\phi_j)\otimes(\epsilon\phi_k)\\ \phi_j\otimes\phi_k & \phi_j\otimes(\epsilon\phi_k)\end{pmatrix}+\begin{pmatrix}0 & \epsilon\\ 0 & 0\end{pmatrix}. \tag{18}$$

Let us denote by $\widetilde{\mathscr{L}}_I$ the following kernel

$$\widetilde{\mathscr{L}}_I(x,y)=\sum_{1\leqslant j,k\leqslant 2n} (M^{X\setminus I})^{-t}_{jk}\begin{pmatrix} -(\epsilon_{X\setminus I}\phi_j)\otimes\phi_k & -(\epsilon_{X\setminus I}\phi_j)\otimes(\epsilon_{X\setminus I}\phi_k)\\ \phi_j\otimes\phi_k & \phi_j\otimes(\epsilon_{X\setminus I}\phi_k)\end{pmatrix} +\begin{pmatrix}0 & \epsilon\\ 0 & 0\end{pmatrix}. \tag{19}$$

As above, $\epsilon\phi$ stands for $\int_X \epsilon(x, y)\, \phi(y)$. We use the notation $\phi_j \otimes \phi_k$ as a shorthand for $\phi_j(x)\, \phi_k(y)$. To prove the main result of the paper we will show that $\tilde{\mathcal{L}}_I = \mathcal{K}_I(\mathrm{Id} - \mathcal{K}_I)^{-1}$ (in other words we are going to prove that $\tilde{\mathcal{L}}_I = \mathcal{L}_I$, where $\mathcal{L}_I$ is defined in (17)). The proof relies on Lemmas 1 and 2 given below. Let us introduce the notation $(\epsilon_I \phi)(x) = \int_I \epsilon(x, y)\, \phi_s(y)\, d\lambda(y)$. We will show that the finite-dimensional subspace $\mathcal{H} = \mathrm{Span}\{\binom{\epsilon\phi_s}{-\phi_s}, \binom{-\epsilon\phi_s}{-\phi_s}, \binom{\epsilon_I\phi_s}{0}\}_{s=1,\dots 2n}$ is invariant under $\mathcal{K}_I$ and $\tilde{\mathcal{L}}_I$. The main part of the proof of the theorem is to show that $\tilde{\mathcal{L}}_I = \mathcal{K}_I(\mathrm{Id} - \mathcal{K}_I)^{-1}$ holds on $\mathcal{H}$.

Lemma 3.1. The operators $\mathcal{K}_I$, $\tilde{\mathcal{L}}_I$ leave $\mathcal{H}$ invariant and $\tilde{\mathcal{L}}_I = \mathcal{K}_I(\mathrm{Id} - \mathcal{K}_I)^{-1}$ holds on $\mathcal{H}$.

Below we give the proof of the lemma. Using the notations introduced above in (8)–(10) one can easily calculate

$$\mathcal{K}_I \begin{pmatrix} \epsilon\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots,2n} -((G^I)^t M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} \tag{20}$$

$$\mathcal{K}_I \begin{pmatrix} 0 \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots,2n} (G^I M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}. \tag{21}$$

Defining the $2n \times 2n$ matrix T as

$$T_{sk} = \int_I \phi_s(x) \int_{X\setminus I} \epsilon(x, y)\, \phi_k(y)\, d\lambda(y)\, d\lambda(x) \tag{22}$$

we compute

$$\mathcal{K}_I \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots 2n} ((G^I - T)\, M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}, \tag{23}$$

where $(G^I - T)_{sk} = M^I_{sk} = \int_{I^2} \phi_s(x)\, \epsilon(x, y)\, \phi_k(y)\, \lambda(dx)\, \lambda(dy)$. One can rewrite Eqs. (20) and (21) as

$$\mathcal{K}_I \begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n} ((G^I - (G^I)^t)\, M^{-1})^t_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, \tag{24}$$

$$\mathcal{K}_I \begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n} ((G^I + (G^I)^t)\, M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}. \tag{25}$$

We conclude that that the subspace $\mathscr{H}$ is indeed invariant under $\mathscr{K}_I$ and the matrix of the restriction of $\mathscr{K}_I$ on $\mathscr{H}$ has the following block structure in the basis $\{\binom{\epsilon\phi_s}{-\phi_s}, \binom{-\epsilon\phi_s}{-\phi_s}, \binom{\epsilon_I\phi_s}{0}\}_{s=1,\dots 2n}$:

$$\begin{pmatrix} (G^I-(G^I)^t)\,M^{-1} & (G^I+(G^I)^t)\,M^{-1} & (G^I-T)\,M^{-1} \\ 0 & 0 & 0 \\ -\mathrm{Id} & -\mathrm{Id} & 0 \end{pmatrix} \tag{26}$$

(in particular $\mathrm{Ran}(\mathscr{K}_I|_{\mathscr{H}}) = \mathrm{Span}\{\binom{\epsilon\phi_s}{-\phi_s}, \binom{\epsilon_I\phi_s}{0}\}_{s=1,\dots 2n}$). Let us introduce some additional notations:

$$A = (G^I-(G^I)^t)\,M^{-1}, \tag{27}$$

$$B = (G^I+(G^I)^t)\,M^{-1}, \tag{28}$$

$$C = (G^I-T)\,M^{-1}. \tag{29}$$

When a matrix has a block form

$$\mathscr{M} = \begin{pmatrix} A & B & C \\ 0 & 0 & 0 \\ -\mathrm{Id} & -\mathrm{Id} & 0 \end{pmatrix}$$

(as it is in our case) the matrix $\mathscr{M}(\mathrm{Id}-\mathscr{M})^{-1}$ has the block form

$$\begin{pmatrix} (\mathrm{Id}-A+C)^{-1}-\mathrm{Id} & (B-C)(\mathrm{Id}-A+C)^{-1} & C(\mathrm{Id}-A+C)^{-1} \\ 0 & 0 & 0 \\ -(\mathrm{Id}-A+C)^{-1} & -\mathrm{Id}-(B-C)(\mathrm{Id}-A+C)^{-1} & -C(\mathrm{Id}-A+C)^{-1} \end{pmatrix}. \tag{30}$$

As one can see from the formulas (31)–(33) the invertibility of $\mathrm{Id}-\mathscr{M}$ follows from the invertibility of $M^{X\setminus I}$ which has been assumed throughout the paper. We have

$$(\mathrm{Id}-A+C)^{-1} = M(M+(G^I)^t-T)^{-1} = M(M^{X\setminus I})^{-1} \tag{31}$$

$$C(\mathrm{Id}-A+C)^{-1} = (G^I-T)(M+(G^I)^t-T)^{-1} = M^I(M^{X\setminus I})^{-1} \tag{32}$$

$$\begin{aligned}(B-C)(\mathrm{Id}-A+C)^{-1} &= ((G^I)^t+T)(M+(G^I)^t-T)^{-1} \\ &= ((G^I)^t+T)(M^{X\setminus I})^{-1}. \end{aligned} \tag{33}$$

Let us now compute the matrix of the restriction of $\tilde{\mathcal{L}}_I$ on $\mathcal{H}$. We have

$$\tilde{\mathcal{L}}_I = \sum_{j,k=1,\dots,2n} (M^{X\setminus I}_{jk})^t \begin{pmatrix} -(\epsilon_{X\setminus I}\phi_j)\otimes\phi_k & -(\epsilon_{X\setminus I}\phi_j)\otimes(\epsilon_{X\setminus I}\phi_k) \\ \phi_k\otimes\phi_k & \phi_j\otimes(\epsilon_{X\setminus I}\phi_k) \end{pmatrix} + \begin{pmatrix} 0 & \epsilon_{X\setminus I} \\ 0 & 0 \end{pmatrix}. \tag{34}$$

Similarly to the computations above one can see that $\mathcal{H}$ is invariant under $\tilde{\mathcal{L}}_I$ and

$$\tilde{\mathcal{L}}_I \begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n} ((T-(G^I)^t)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}$$
$$- \sum_{1\leqslant j\leqslant 2n} ((T-(G^I)^t)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, \tag{35}$$

$$\tilde{\mathcal{L}}_I \begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n} ((T+(G^I)^t)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}$$
$$- \sum_{1\leqslant j\leqslant 2n} ((T+(G^I)^t)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, \tag{36}$$

and

$$\tilde{\mathcal{L}}_I \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots,2n} ((G^I-T)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}$$
$$- \sum_{j=1,\dots,2n} ((G^I-T)(M^{X\setminus I})^{-1})_{sj} \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix}. \tag{37}$$

Therefore the restriction of $\tilde{\mathcal{L}}_I$ to $\mathcal{H}$ in the basis $\{\binom{\epsilon\phi_s}{-\phi_s}, \binom{-\epsilon\phi_s}{-\phi_s}, \binom{\epsilon_I\phi_s}{0}\}_{s=1,\dots 2n}$ has the following block structure

$$\begin{pmatrix} (T-(G^I)^t)(M^{X\setminus I})^{-1} & (T+(G^I)^t)(M^{X\setminus I})^{-1} & (G^I-T)(M^{X\setminus I})^{-1} \\ 0 & 0 & 0 \\ -\mathrm{Id}-(T-(G^I)^t)(M^{X\setminus I})^{-1} & -\mathrm{Id}-(T+(G^I)^t)(M^{X\setminus I})^{-1} & -(G^I-T)(M^{X\setminus I})^{-1} \end{pmatrix}. \tag{38}$$

Comparing (30), (31)–(33), and (38) we see that $\tilde{\mathcal{L}}_I = \mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}$ on $\mathcal{H}$. Lemma 3.1 is proven.

To show that $\widetilde{\mathscr{L}}_I = \mathscr{K}_I(\mathrm{Id} - \mathscr{K}_I)^{-1}$ also holds on the complement of $\mathscr{H}$ it is enough to prove it on the subspaces $\binom{(\mathscr{H}_1)^\perp}{0}$, and $\binom{0}{(\mathscr{H}_2)^\perp}$, where $\mathscr{H}_1 = \mathrm{Span}(\overline{\epsilon_I \phi_s})_{k=1,\dots,2n}$ and $\mathscr{H}_2 = \mathrm{Span}(\overline{\phi_s})_{k=1,\dots,2n}$. The invertibility of the matrix M_I implies that actually it is enough to prove $\widetilde{\mathscr{L}}_I = \mathscr{K}_I(\mathrm{Id} - \mathscr{K}_I)^{-1}$ on the subspaces $\binom{(\mathscr{H}_2)^\perp}{0}$, and $\binom{0}{(\mathscr{H}_1)^\perp}$. Here we use the standard notation $(\mathscr{H}_i)^\perp$ for the orthogonal complement in $L^2(I)$ with the standard scalar product $(f, g)_I = \int_I \overline{f(x)}\, g(x)\, d\lambda(x)$. We start with the first subspace.

Lemma 3.2. The relation $\widetilde{\mathscr{L}}_I = \mathscr{K}_I(\mathrm{Id} - \mathscr{K}_I)^{-1}$ holds on $\binom{0}{(\mathscr{H}_1)^\perp}$.

The proof is a straightforward check. The notations are slightly simplified when the functions $\{\epsilon_I \phi_k, \epsilon \phi_k, k = 1,\dots, 2n\}$ are linearly independent in $L^2(I)$. The degenerate case is left to the reader. Consider $f_s \in (\mathscr{H}_1)^\perp$, $s = 1,\dots, 2n$ such that

$$(\overline{\epsilon\phi_k}, f_s)_I = (\overline{\epsilon\phi_k}, \phi_s)_I, \qquad k = 1,\dots, 2n. \tag{39}$$

We are going to establish the relation for $\binom{0}{f_s}$, which then immediately extends by linearity to the linear combinations of $\binom{0}{f_s}$. We write

$$\begin{aligned}
\mathscr{K}_I \begin{pmatrix} 0 \\ -f_s \end{pmatrix} &= \sum_{j,k=1,\dots,2n} M^{-t}_{jk} (\overline{\epsilon\phi_k}, -f_s)_I \begin{pmatrix} -\epsilon\phi_j \\ \phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} \\
&= \sum_{j,k=1,\dots,2n} M^{-t}_{jk} (\overline{\epsilon\phi_k}, -\phi_s)_I \begin{pmatrix} -\epsilon\phi_j \\ \phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} \\
&= \sum_{j=1,\dots,2n} (G^I M^{-1}_{sj}) \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}
\end{aligned} \tag{40}$$

(we have used (39) in the second equality) and

$$\mathscr{K}_I \begin{pmatrix} \epsilon_I \phi_s \\ -f_s \end{pmatrix} = \sum_{j=1,\dots,2n} ((G^I - (G^I)^t)\, M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}. \tag{41}$$

Combining (40) and (41) we get

$$\mathscr{K}_I \begin{pmatrix} -\epsilon_I \phi_s \\ -f_s \end{pmatrix} = \sum_{j=1,\dots,2n} ((G^I + (G^I)^t)\, M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}. \tag{42}$$

Similarly to (23) we compute

$$\mathscr{K}_I \begin{pmatrix} \epsilon_I \phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots,2n} ((G^I - T)\, M^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}. \tag{43}$$

It should be noted that $\mathcal{K}_I\binom{\epsilon_I f_s}{0}=0$ because $\int_I (\epsilon_I\phi_s)(x)\,\phi_j(x)\,d\lambda(x)=-\int_I f_s(x)(\epsilon_I\phi_j)(x)\,d\lambda(x)=0$ for all $j=1,\dots,2n$. This together with (39) allows us to conclude that the calculation of $\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\binom{0}{f_s}$ is almost identical to the calculation of $\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\binom{0}{\phi_s}$ with the only difference that in the former one we have to replace the term $-\binom{\epsilon_I\phi_s}{0}$ by $-\binom{\epsilon_I f_s}{0}$ (see the last equation of (44)). Namely

$$\begin{aligned}
&\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\begin{pmatrix}0\\-f_s\end{pmatrix}\\
&\quad=\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\left(\frac{1}{2}\begin{pmatrix}\epsilon\phi_s\\-f_s\end{pmatrix}+\begin{pmatrix}-\epsilon\phi_s\\-f_s\end{pmatrix}\right)\\
&\quad=\sum_{j=1,\dots,2n}(1/2)((A+B)(\mathrm{Id}-A+C)^{-1})_{sj}\begin{pmatrix}\epsilon_I\phi_j\\-\phi_j\end{pmatrix}\\
&\qquad-\sum_{j=1,\dots,2n}(1/2)((A+B)(\mathrm{Id}-A+C)^{-1})_{sj}\begin{pmatrix}\epsilon_I\phi_j\\0\end{pmatrix}-\begin{pmatrix}\epsilon_I f_s\\0\end{pmatrix}\\
&\quad=\sum_{j=1,\dots,2n}[G^I(M^{X\setminus I})^{-1}]_{sj}\left[\begin{pmatrix}\epsilon\phi_j\\-\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_I\phi_j\\0\end{pmatrix}\right]-\begin{pmatrix}\epsilon_I f_s\\0\end{pmatrix}.
\end{aligned}\tag{44}$$

where A, B, C are defined in (27)–(29). At the same time

$$\begin{aligned}
\widetilde{\mathcal{L}}_I\begin{pmatrix}0\\-f_s\end{pmatrix}&=\sum_{j,k=1,\dots,2n}((M^{X\setminus I})^{-t})_{jk}\begin{pmatrix}\epsilon_{X\setminus I}\phi_j\\-\phi_j\end{pmatrix}(\overline{\epsilon_{X\setminus I}\phi_k},f_s)_I-\begin{pmatrix}\epsilon_I f_s\\0\end{pmatrix}\\
&=\sum_{j,k=1,\dots,2n}((M^{X\setminus I})^{-t})_{jk}\begin{pmatrix}\epsilon_{X\setminus I}\phi_j\\-\phi_j\end{pmatrix}(\overline{\epsilon\phi_k},f_s)_I-\begin{pmatrix}\epsilon_I f_s\\0\end{pmatrix}\\
&=[G^I(M^{X\setminus I})^{-1}]_{sj}\left[\begin{pmatrix}\epsilon\phi_j\\-\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_I\phi_j\\0\end{pmatrix}\right]-\begin{pmatrix}\epsilon_I f_s\\0\end{pmatrix}.
\end{aligned}\tag{45}$$

Therefore $\widetilde{\mathcal{L}}_I\binom{0}{-f_s}=\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\binom{0}{-f_s}$, $s=1,\dots 2n$. By linearity result follows for all $\binom{0}{f}$ such that $(\overline{\epsilon_I\phi_k},f)_I=\int_I(\epsilon_I\phi_k)(x)\,f(x)\,d\lambda(x)=0$, $k,j=1,\dots,2n$. Lemma 3.2 is proven.

To check (17) on $\binom{(\mathcal{H}_2)^{\perp}}{0}$ we note that $\mathcal{K}_I(\mathrm{Id}-\mathcal{K}_I)^{-1}\binom{g}{0}=\widetilde{\mathcal{L}}_I\binom{g}{0}=0$ for g such that $\int_I g(x)\,\phi_k(x)\,d\lambda(x)=0$, $k=1,\dots,2n$, which together with the invertibility of M finishes the proof. The theorem is proven.

ACKNOWLEDGMENTS

It is a great pleasure to thank Alexei Borodin and Eric Rains for useful discussions. The author also would like to thank the anonymous referee for pointing out a number of misprints. Research was supported in part by the Sloan Research Fellowship and NSF Grant DMS-0103948.

REFERENCES

1. A. Borodin, Biorthogonal ensembles, *Nucl. Phys. B* **536**:704–732 (1999).
2. A. Borodin and G. Olshanski, Distributions on partitions, point processes, and the hypergeometric kernel, *Commun. Math. Phys.* **211**:335–358 (2000).
3. A. Borodin and A. Soshnikov, Janossy densities. I. Determinantal ensembles, *J. Stat. Phys.* **113**:595–611 (2003).
4. D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes* (Springer-Verlag, New York, 1988).
5. C. D. Godsil, *Algebraic Combinatorics* (Chapman & Hall, New York, 1993).
6. M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
7. K. A. Muttalib, Random matrix models with additional interactions, *J. Phys. A* **285**:L159 (1995).
8. G. Olshanski, Pfaffian processes, unpublished note (2002).
9. E. Rains, Correlation functions for symmetrized increasing subsequences, available at arXiv:math.CO/0006097.
10. A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys* **55**:923–975 (2000).
11. C. A. Tracy and H. Widom, Correlation functions, Cluster functions, and spacing distributions for random matrices, *J. Stat. Phys.* **92**:809–835 (1998).
12. H. Widom, On the relation between orthogonal, symplectic and unitary matrix ensembles, *J. Stat. Phys.* **94**:347–363 (1999).